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Casimir effect across a layered medium

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Using nonstandard recursion relations for Fresnel coefficients involving successive stacks of layers, we extend the Lifshitz formula to configurations with an inhomogeneous, n-layered, medium separating two planar objects. The force on each object is the sum of a Lifshitz like force and a force arising from the inhomogeneity of the medium. The theory correctly reproduces very recently obtained results for the Casimir force/energy in some simple systems of this kind. As a by product, we obtain a formula for the force on an (unspecified) stack of layers between two planar objects which generalizes our previous result for the force on a slab in a planar cavity.

Keywords: Casimir force; Fresnel coefficients; nonstandard recursion.

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1. Introduction

Very recently, several papers appeared dealing with the theory of the Casimir effect in systems consisting of two perfectly reflecting plates separated by a layered medium[1,2,3]. Among the other results, these works provided formulas for the Casimir force and/or energy for a few simple systems of this sort (with up to five layers media[1] between the plates). Using the theory of the Casimir effect in multilayers[4] and nonstandard recursion relations for Fresnel coefficients[4,5,6,7], in this Note we derive formulas for the Casimir force and energy for systems with arbitrary plates separated by arbitrary inhomogeneous, generally n-layered, media.

2. Casimir effect across a layered medium

Consider the system consisting of two planar objects (plates) separated by a layered medium, as depicted in Fig. 1. According to the theory of the Casimir effect in multilayers[4], the Casimir forces on the left (L) and the right (R) plate are given by

$$F_L \equiv F_{1-} = T_{zz}^{(1)} \quad \text{and} \quad F_R \equiv F_{n+} = -T_{zz}^{(n)}, \tag{1}$$

respectively, where (unless necessary, we omit the polarization index $q = p, s$ when writing Fresnel coefficients)

$$T_{zz}^{(j)} = \frac{\hbar}{2\pi^2} \int_0^\infty d\xi \int_0^\infty dk k \kappa_j \sum_q \frac{r_{j-} r_{j+} e^{-2\kappa_j d_j}}{1 - r_{j-} r_{j+} e^{-2\kappa_j d_j}} \tag{2}$$

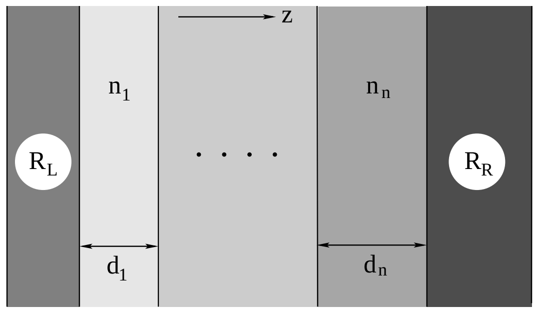

Fig. 1. Two plates separated by a layered medium shown schematically. Plates are described by their reflection coefficients R_L and R_R and the medium layers by their (complex) refraction indexes $n_a(\omega) = \sqrt{\varepsilon_a(\omega)\mu_a(\omega)}$, $a = 1 \dots n$.

is the relevant component of the vacuum-field (Minkowski) stress tensor in the layer j. Here $\kappa_j = \sqrt{n_j^2(i\xi)\xi^2/c^2 + k^2}$ is the perpendicular wave vector at the imaginary frequency ($\omega = i\xi$) in the layer, $k = \sqrt{k_x^2 + k_y^2}$ is the magnitude of the wave vector parallel to the system surfaces and $r_{j\pm}(i\xi, k)$ are the reflection coefficients of the right and left stack of layers bounding the layer j. These reflection coefficients obey generalized recursion relations [4,5,6,7]

$$r_{j\pm} = r_{j/l} + \frac{t_{j/l}t_{l/j}r_{l\pm}e^{-2\kappa_l d_l}}{1 - r_{l/j}r_{l\pm}e^{-2\kappa_l d_l}}, \quad t_{j\pm} = \frac{t_{j/l}t_{l\pm}e^{-\kappa_l d_l}}{1 - r_{l/j}r_{l\pm}e^{-2\kappa_l d_l}}, \tag{3}$$

where l denotes an intermediate layer and where the symbol $a/b \equiv a \dots b$ is used to denote the stack of layers between layers a and b. As seen, these recurrence relations look the same as the standard ones [8,9] (to which they reduce in case that layers j and l are neighbor layers), however, this time they generally involve Fresnel coefficients $r_{j/l}$, $r_{l/j}$, $t_{j/l}$ and $t_{l/j}$ of the *stack* between the layers j and l.

Using the above recursion relations, reflection coefficients r_{1+} and r_{n-} can be expressed as

$$r_{1+} = \frac{r_{1/n} + a_{1/n}R_R e^{-2\kappa_n d_n}}{1 - r_{n/1}R_R e^{-2\kappa_n d_n}}, \quad r_{n-} = \frac{r_{n/1} + a_{n/1}R_L e^{-2\kappa_1 d_1}}{1 - r_{1/n}R_L e^{-2\kappa_1 d_1}}, \tag{4}$$

where we have introduced the quantity

$$a_{1/n} = t_{1/n}t_{n/1} - r_{1/n}r_{n/1} = a_{n/1} \tag{5}$$

and identified r_{n+} and r_{1-} as the reflection coefficients R_R and R_L, respectively, of the plates. Therefore, from (1) and (2) we obtain for the forces on the plates

$$F_L = \frac{\hbar}{2\pi^2}\int_0^\infty d\xi \int_0^\infty dkk\kappa_1 \sum_q \frac{1}{N_n}(r_{1/n} + a_{1/n}R_R e^{-2\kappa_n d_n})R_L e^{-2\kappa_1 d_1}, \tag{6a}$$

$$F_R = \frac{-\hbar}{2\pi^2}\int_0^\infty d\xi \int_0^\infty dkk\kappa_n \sum_q \frac{1}{N_n}(r_{n/1} + a_{1/n}R_L e^{-2\kappa_1 d_1})R_R e^{-2\kappa_n d_n}, \tag{6b}$$

where

$$N_n = 1 - (r_{1/n} R_L e^{-2\kappa_1 d_1} + r_{n/1} R_R e^{-2\kappa_n d_n}) - a_{1/n} R_L R_R e^{-2\kappa_1 d_1 - 2\kappa_n d_n}. \quad (7)$$

As seen, since $r_{1/n} \neq r_{n/1}$ unless the medium is not symmetric across the gap, these forces are not generally equal in magnitude and each of them consists of a Lifshitz-like force (given by the second terms in (6)) and a force due to the inhomogeneity of the medium. We also note that owing to the medium inhomogeneity there is a force[4] $F_S = T_{zz}^{(n)} - T_{zz}^{(1)} = -F_R - F_L$ on the central stack of the medium given explicitly by

$$F_S = \frac{\hbar}{2\pi^2} \int_0^\infty d\xi \int_0^\infty dkk \sum_q \frac{1}{N_n} (\kappa_n r_{n/1} R_R e^{-2\kappa_n d_n} - \kappa_1 r_{1/n} R_L e^{-2\kappa_1 d_1})$$
$$+ (\kappa_n - \kappa_1) \frac{a_{1/n}}{N_n} R_L R_R e^{-2\kappa_1 d_1 - 2\kappa_n d_n}. \quad (8)$$

When $n_1 = n_n$, this generalizes previously obtained result for the force on a slab in a planar cavity[4,7,10] to configurations with the slab replaced by an unspecified multilayered stack.

Having determined forces on the plates, we can calculate the Casimir energy of the system from[4] $F_L = \partial E/\partial d_1$ or $F_R = -\partial E/\partial d_n$. We obtain

$$E = \frac{\hbar}{(2\pi)^2} \int_0^\infty d\xi \int_0^\infty dkk \sum_q \ln N_n. \quad (9)$$

In the following, we illustrate this formula by discussing its implications for several simple systems and comparing them with the results obtained in Refs. 1-3.

3. Discussion

It is easy to see that (7) and (9) give correctly the Casimir energy in case of a homogeneous medium between the plates. Indeed, assuming that all n medium layers are made of the same matter ($n_a = n$, $\kappa_a = \kappa$), we have $r_{1/n} = r_{n/1} = 0$ and $t_{1/n} = t_{n/1} = \exp[-\kappa(d - d_1 - d_n)]$ so that $a_{1/n} = \exp[-2\kappa(d - d_1 - d_n)]$ and

$$N_n = 1 - R_L R_R e^{-2\kappa d}, \quad (10)$$

where d is the distance between the plates. This leads to the standard Lifshitz-type formula[11] for the Casimir energy. For perfectly reflecting plates[2,3], we must let here $R_L R_R = 1$.

In the $n = 2$ case (two media between the plates), we have $r_{1/2} = r_{12} = -r_{21} = -r_{2/1}$ and $t_{1/2} = t_{12} = (\mu_2\kappa_1/\mu_1\kappa_2)t_{21} = (\mu_2\kappa_1/\mu_1\kappa_2)t_{2/1}$, where r_{12} and t_{12} are the *single-interface* Fresnel coefficients[4,5]

$$r_{12} = \frac{\kappa_1 - \gamma_{12}\kappa_2}{\kappa_1 + \gamma_{12}\kappa_2} = -r_{21}, \quad t_{12} = \sqrt{\frac{\gamma_{12}}{\gamma_{12}^s}}(1 + r_{12}) = \frac{\mu_2\kappa_1}{\mu_1\kappa_2} t_{21}, \quad (11)$$

with $\gamma_{12}^p = \varepsilon_1/\varepsilon_2$ and $\gamma_{12}^s = \mu_1/\mu_2$. Noting that $a_{1/2} = 1$, we have

$$N_2 = 1 - r_{12}(R_L e^{-2\kappa_1 d_1} - R_R e^{-2\kappa_2 d_2}) - R_L R_R e^{-2(\kappa_1 d_1 + \kappa_2 d_2)}, \quad (12)$$

which, in conjunction with (9), gives the Casimir energy for the present system. This result coincides with the corresponding result obtained in Ref. 2 providing that we let for perfectly reflecting plates $R^q_{L(R)} = -1$. We note, however, that perfect reflectors are standardly simulated by media with infinitely large permittivities (conductivities) in which case (11) implies ($\varepsilon_2 \to \infty$) that $R^s_{L(R)} = -1$ but $R^p_{L(R)} = 1$. Therefore, with this convention, our result disagrees with that of Ref. 2 regarding the p contribution to the Casimir force/energy.

Using recursion relations (3), E for more complex ($n \geq 3$) systems can be written in terms of lower-layered stacks and, owing to the number of the medium layers, this can be done in a number of ways. Clearly, to obtain the effective Casimir energy, we can drop from these results the terms not involving d_1 or d_n. Thus, for example, from (3) we have[7]

$$r_{1/n} = \frac{r_{1/l} + a_{1/l} r_{l/n} e^{-2\kappa_l d_l}}{D_l}, \qquad r_{n/1} = \frac{r_{n/l} + a_{n/l} r_{l/1} e^{-2\kappa_l d_l}}{D_l}, \tag{13a}$$

$$a_{1/n} = \frac{a_{1/l} a_{n/l} e^{-2\kappa_l d_l} - r_{1/l} r_{n/l}}{D_l}, \qquad D_l = 1 - r_{l/1} r_{l/n} e^{-2\kappa_l d_l}. \tag{13b}$$

Using this in (7) and rearranging, we find

$$N_n = \frac{N_n^{(l)}}{D_l}, \tag{14}$$

where

$$N_n^{(l)} = (1 - r_{1/l} R_L e^{-2\kappa_1 d_1})(1 - r_{n/l} R_R e^{-2\kappa_n d_n}) \\ - e^{-2\kappa_l d_l}(a_{1/l} R_L e^{-2\kappa_1 d_1} + r_{l/1})(a_{n/l} R_R e^{-2\kappa_n d_n} + r_{l/n}). \tag{15}$$

Finally, inserting this N_n in (9) and dropping the (ineffective) term involving D_l, we find for the effective Casimir energy E_l of the system (with respect to the layer l)

$$E_l = \frac{\hbar}{(2\pi)^2} \int_0^\infty d\xi \int_0^\infty dkk \sum_q \ln N_n^{(l)}. \tag{16}$$

This generalizes the ($T = 0$) result for the Casimir interaction energy between two slabs obtained in Ref 1 using a realistic Casimir piston approach and a five layer model for the medium to arbitrary multilayered slabs and plates. Note that, when removing the plates by letting $d_{1(n)} \to \infty$, we have $N_n^{(l)} \to D_l$ and (15) and (16) give the Casimir interaction energy of the two stacks of layers separated by a layer of medium l, as derived in Ref. 4.

We illustrate the above result by considering the n=3 system. In this case, there is only one intermediate layer and the effective Casimir energy (16) is entirely expressed in terms of the single-interface reflection coefficients $r_{12} = -r_{21}$ and $r_{32} = -r_{23}$. From (15), we have ($a_{1/2} = a_{3/2} = 1$)

$$N_3^{(2)} = (1 - r_{12} R_L e^{-2\kappa_1 d_1})(1 - r_{32} R_R e^{-2\kappa_3 d_3}) \\ - e^{-2\kappa_2 d_2}(R_L e^{-2\kappa_1 d_1} - r_{12})(R_R e^{-2\kappa_3 d_3} - r_{32}). \tag{17}$$

As mentioned, for perfectly reflecting plates we must let here $R^q_{L(R)} = \delta_{qp} - \delta_{qs}$. This result then coincides with the corresponding result derived in Ref. 1 whereas the results obtained in Refs. 2 and 3 correspond to plates with $R^q_{L(R)} = -1$ and $R^q_{L(R)} = 1$, respectively.

4. Summary

Effectively, in this work we have extended the Lifshitz formula to configurations with an inhomogeneous, n-layered, medium separating two planar objects. The force on each object is the sum of a Lifshitz-like force and a force arising from the inhomogeneity of the medium. Owing to this inhomogeneity, there is also a force acting on the medium. When the first and the last medium layer are made of the same matter, this result generalizes previously obtained one for the force on a slab in a planar cavity to arbitrary multilayered slabs.

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References

1. L. P. Teo, Phys. Rev. A **81**, 032502 (2010).
2. F. Kheirandish, M. Soltani and J. Sarabadani, Ann. Phys. **326**, 657 (2011).
3. E. Amooghorban, M. Wubs, N. A. Mortensen and F. Kheirandish, Phys. Rev. A **84**, 013806 (2011).
4. M. S. Tomaš, Phys. Rev. A **66**, 052103 (2002).
5. M. S. Tomaš, Phys. Rev. A **51**, 2545 (1995).
6. C. C. Katsidis and D. I. Siapkas, Appl. Opt. **41**, 3978 (2002).
7. M. S. Tomaš, Phys. Rev. A **81**, 044104 (2010). Note that the last term in (25) is of the wrong sign.
8. M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1980).
9. W. C. Chew, *Waves and Fields in Inhomogeneous Media* (IEEE, New York, 1995).
10. S. A. Ellingsen and I. Brevik, J. Phys. A: Math. Theor. **40**, 3643 (2007).
11. I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, Adv. Phys. **10**, 165 (1961).